Annals of Pure and Applied Logic 34 (1987) 87–97
North-Holland


# EXTRASPECIAL $p$-GROUPS

Saharon SHELAH
*Institute of Mathematics, Hebrew University, Jerusalem, Israel*

Juris STEPRĀNS
*York University, 4700 Keele St., Downsview, Ontario, M3J 1P3 Canada*

Communicated by A. Nerode
Received 7 August 1985

It is shown that there is a class of cardinals $\lambda$ for which there are $2^\lambda$ extraspecial $p$-groups of size $\lambda$ without abelian subgroups of size $\lambda$. This improves results of Ehrenfeucht and Faber and answers questions of Tomkinson.

## 1. Introduction

In his survey of FC-groups [7] Tomkinson poses some questions about extraspecial $p$-groups which will be answered in this paper. Ehrenfeucht and Faber used the generalized continuum hypothesis (GCH) to construct extraspecial $p$-groups all of whose abelian subgroups have cardinality smaller than that of the group [2]. A result of Erdös suggested to them that the GCH is probably not necessary for their theorem. It will be shown that this is indeed the case and that it is even possible to construct such a group of size $\aleph_1$. Tomkinson also asks if it is possible to construct $2^\lambda$ non-isomorphic extraspecial $p$-groups of size $\lambda$. Without assuming any extra set-theoretic hypotheses it will be shown that the answer is yes for a proper class of cardinals.

The main difficulty in constructing such groups is in not introducing large abelian subgroups. In fact if one is not concerned with the size of abelian subgroups the result follows from the fact that the theory of extraspecial $p$-groups is unstable [6].

Both results will be obtained from a general framework which, among other things, will include the result of Ehrenfeucht and Faber. The advantage of this approach is that it will isolate the set-theoretic combinatorics from the group theory. The combinatorics developed in this paper are extensions of results in [5]. These, in turn, were motivated by Todorčević's denial of the partition relation $\omega_1 \to [\omega_1]^2_{\aleph_1}$.

Before the details can be explained, some definitions and notation are necessary. An extraspecial $p$-group is a group $G$ such that

(1) $G' = Z(G)$ (the derived group coincides with the centre).
(2) $G'$ is isomorphic to $\mathbb{Z}_p$.
(3) $G/G'$ is isomorphic to the direct product of copies of $\mathbb{Z}_p$.

The term Ehrenfeucht–Faber group will refer to an extraspecial $p$-group of cardinality $\lambda$ which has no abelian subgroups of size $\lambda$.

Ordinals will be considered as the set of their predecessors even in the case of finite ordinals. So, for example, $4 = \{0, 1, 2, 3\}$ and $\alpha \in \beta$ means the same thing as $\alpha < \beta$. The first infinite ordinal is $\omega$. The following notation will also be used:

(a) $|X|$ is the cardinality of $X$.
(b) ${}^XY$ is the set of function from $X$ to $Y$.
(c) $[X]^\lambda$ is the set of subsets of $X$ of size $\lambda$.
(d) $[X]^{<\lambda}$ and $[X]^{\leq\lambda}$ are defined similarly.
(e) $\chi_A$ is the characteristic function of $A$.
(f) $F''X$ is the image of $X$ under the function $F$.

Finally, if $A$ is a set of ordinals, then $A(i)$ will refer to the $i$th element of $A$ under the inherited ordering.

The following well known fact, known as the delta system lemma, will prove to be useful: If $\mathcal{A}$ is a family of finite sets and $|\mathcal{A}|$ is regular, then there is $\mathcal{B} \subseteq \mathcal{A}$ and a finite set $R$ such that $|\mathcal{B}| = |\mathcal{A}|$ and such that $A \cap B = R$ for every pair $\{A, B\} \in [\mathcal{B}]^2$. The family $\mathcal{B}$ is known as a delta system with root $R$. For a proof consult either Kunen [3] or Williams [8].

It is worth noting that the methods developed in this paper can be used to handle different types of problems as well. For example, it is possible to construct, under CH, an Ehrenfeucht–Faber group in which there are no two commuting subgroups of the same cardinality as the group. This might be seen as a group-theoretic version of the partition relation $\lambda \to (\lambda; \lambda)^2$.

## 2. The combinatorics

Let $X \in [\lambda]^\lambda$, $\Phi : [X]^2 \to 2$ be an arbitrary function and $A$ and $B$ be disjoint finite subsets of $X$. Then $M^\Phi_{A,B}$ is the 2-valued $|A| \times |B|$ matrix defined by $M^\Phi_{A,B}(i, j) = \Phi(\{A(i), B(j)\})$. If $\mathcal{M}$ is a set of 2-valued $k \times k$ matrices, then $\Phi$ will be said to $\mu$-realize $\mathcal{M}$ if whenever $\{A_\xi : \xi \in \mu\} \subseteq [\lambda]^k$ is a sequence of disjoint sets, then there exist $\xi \in \eta \in \mu$ such that $M^\Phi_{A_\xi, A_\eta} \in \mathcal{M}$.

Finding functions $\Phi$ which $\mu$-realize the appropriate sets of matrices is the key to constructing Ehrenfeucht–Faber groups. The precise meaning of 'the appropriate sets of matrices' will depend on the theorem to be proved. For the purposes of this paper the meaning of this phrase can be restricted to the conclusions of the following theorems. However, it is to be hoped that a more careful analysis of the combinatorial requirements of extraspecial $p$-groups might lead to more subtle theorems.

**Theorem 1.** *Assuming* $2^{\aleph_0} = \aleph_1$ *there is a function* $\Phi : [\omega_1]^2 \to 2$ *which* $\omega_1$*-realizes every singleton.*

**Theorem 2.** $\mathrm{MA}_{\aleph_1}$ *implies that there is no function* $\Phi : [\omega_1]^2 \to 2$ *which* $\omega_1$*-realizes every singleton.*

**Theorem 3.** *It is consistent (relative to the consistency of* ZF) *that* $2^{\aleph_0} > \aleph_1$ *and there is a function* $\Phi : [\omega_1]^2 \to 2$ *which* $\omega_1$*-realizes every singleton.*

**Theorem 4.** *For any integer k it is consistent that* $\mathrm{MA}_{\aleph_1}$ *holds and every singleton consisting of an* $m \times m$ *2-valued matrix is realized provided that* $m < k$.

**Theorem 5.** *It is consistent that there is a function* $\Phi : [\omega_2]^2 \to 2$ *which* $\omega_1$*-realizes every singleton.*

**Theorem 6.** *For* $d \in {}^k2$ *let* $\mathcal{M}_d$ *be the set of* $k \times k$ *2-valued matrices M satisfying* $M(i, i) = d(i)$ *for all* $i \in k$. *Let* $\lambda$ *be the successor of a regular cardinal. Then there is a function* $\Phi : [\lambda]^2 \to 2$ *which* $\lambda$*-realizes every* $\mathcal{M}_d$. *Moreover,* $\Phi$ *has the property that if* $\{X_\xi : \xi \in \lambda\}$ *are disjoint members of* $[\lambda]^k$, *then there is* $k \times k$ *matrix M and* $\Lambda \in [\lambda]^\lambda$ *such that if* $\{\xi, \eta\} \in [\Lambda]^2$, *then* $M^{\Phi}_{X_\xi, X_\eta}(i, j) = M(i, j)$ *provided that* $i \neq j$. *In fact, the hypothesis on* $\lambda$ *can be weakened to only requiring that* $\lambda$ *has a non-reflecting stationary subset and that there is some* $\tau < \lambda$ *such that* $2^\tau \geq \lambda$.

Since all but the last of these theorems can be proved by standard techniques only an idea of the proofs with appropriate references will be given. For example, Theorem 1 is a standard application of CH. For an example of a similar proof see the original construction of an extraspecial $p$-group by Ehrenfeucht and Faber [2].

The proof of Theorem 2 is similar to Kunen's proof that $\mathrm{MA}_{\aleph_1}$ implies there is no S-space all of whose products are S-spaces [4].

To prove Theorem 3 it suffices to add $\aleph_1$ Cohen reals to a model where $2^{\aleph_0} > \aleph_1$. The Cohen reals can be thought of as adding a function $\Phi : [\omega_1]^2 \to 2$ generically. Standard arguments which can be found in Kunen [3] show that $\Phi$ has the desired property. Adding $\aleph_2$ Cohen reals yields the necessary model to prove Theorem 5.

The proof of Theorem 4 is similar to the proof that $\mathrm{MA}_{\aleph_1}$ is consistent with the existence of a $k$-entangled set which can be found in Avraham–Rubin–Shelah [1].

## 3. The construction

The original construction of an Ehrenfeucht–Faber group used, as an intermediate step, the notion of a symplectic space over a finite field. Although the construction to be presented uses the same ideas the intermediate step will be omitted. It will be shown that for any reasonable function $\Phi:[\lambda]^2 \to 2$ it is possible to construct an extraspecial $p$-group. When extra conditions, such as those in the conclusions of the theorems of the previous section, are imposed on $\Phi$, these translate into extra properties of the associated group.

First, for any subset $X$ of $\lambda$ and any prime $p$ let

$$\bigoplus_X p = \{f \in {}^X p;\ |\{\alpha \in X; f(\alpha) \neq 0\}| < \aleph_0\}.$$

Addition or multiplication of elements of $\bigoplus_X p$ will always refer to the coordinatewise operation modulo $p$. Following Ehrenfeucht and Faber define an operation $*$ on $p \times \bigoplus_X p$ from the function $\Phi$ as follows:

$$(k, f) * (m, g) = \Big(k + m + \sum_{\alpha \in \beta} f(\alpha)\, g(\beta)\, \Phi(\{\alpha, \beta\}), f + g\Big).$$

The arithmetic in the first coordinate is modulo $p$.

It is easy to check that $(p \times \bigoplus_X p, *)$ is a group which is a $p$-group if $p > 2$. This group will be denoted by $G_X$ the dependence on $\Phi$ being understood. A routine calculation reveals that

$$[(m, f), (k, g)] = \Big(-\sum_{\alpha \in \beta} (f(\alpha)\, g(\beta) + g(\alpha) f(\beta))\, \Phi(\{\alpha, \beta\}), 0\Big).$$

Moreover, if every ordinal in the support of $f$ is smaller than every ordinal in the support of $g$ (this phenomenon will be expressed as $f < g$), then this identity reduces to

$$[(m, f), (k, g)] = \Big(-\sum_{\alpha \in \beta} f(\alpha)\, g(\beta)\, \Phi(\{\alpha, \beta\}), 0\Big).$$

Let $\hat{G}_X$ be the subgroup of $G_X\{(i, 0); i \in p\}$. It is clear that $G'_X \subseteq \hat{G}_X$. To get $G'_X = \hat{G}_X$ it suffices to have $\{\alpha, \beta\} \in [X]^2$ such that $\Phi(\{\alpha, \beta\}) = 1$ because then $[(0, \chi_{\{\alpha\}}), (0, \chi_{\{\beta\}})]$ will generate $\hat{G}_X$. It is also easy to see that $\hat{G}_X \subseteq Z(G_X)$. The other inclusion will hold in $G_{X\setminus\xi}$ for some $\xi \in \lambda$ provided that $G_X$ has no abelian subgroup of size $\lambda$. To see this suppose not. Then it is possible to choose $g_\xi \in Z(G_{X\setminus\sigma(\xi)})\setminus\hat{G}_X$ such that $\xi \in \eta$ implies $g_\xi \neq g_\eta$ and that $\sigma(\xi) \in \sigma(\eta)$. Clearly $\{g_\xi : \xi \in \lambda\}$ generate an abelian subgroup of size $\lambda$.

To construct a family of $2^\lambda$ non-isomorphic Ehrenfeucht–Faber groups of size $\lambda$ let $\lambda$ be the successor of a regular cardinal and let $\{X_\xi : \xi \in 2^\lambda\} \subseteq [\lambda]^\lambda$ satisfy the condition that if $\xi \neq \eta$, then $|X_\xi \setminus X_\eta| = \lambda$. It must first be shown that for each $\xi$, $G_{X_\xi}$ has no abelian subgroups of size $\lambda$ where $\Phi$ is provided by Theorem 6.

To see this suppose that $\{(k_\xi, f_\xi); \xi \in \lambda\}$ generate an abelian subgroup. From the delta system lemma it follows that there is $\Lambda \in [\lambda]^\lambda$, $R \in [\lambda]^{<\aleph_0}$ and $k \in \omega$ such that $\{\text{support } (f_\xi); \xi \in \Lambda\}$ form a delta system with root $R$ and that furthermore $|\text{support}(f_\xi)| = k$ for each $\xi \in \Lambda$. It may also be assumed that the restriction of $f_\xi$ to $R$ is the same for all $\xi \in \Lambda$. Hence if $A \in [\Lambda]^p$, then support $(\sum_{\xi \in A} f_\xi) = \bigcup \{\text{support}(f_\xi); \xi \in A\} \setminus R$. Therefore by taking $p$-fold products of distinct elements of $\{(k_\xi, f_\xi); \xi \in \lambda\}$ and reindexing we get an abelian subgroup generated by $\{(m_\xi, g_\xi); \xi \in \lambda\}$ where $\xi \in \eta$ implies $g_\xi < g_\eta$. Let $A_\xi = \text{support } g_\xi$ and $r = p \cdot k - |R| = |A_\xi|$.

From Theorem 6 there is $\Gamma \in [\lambda]^\lambda$ and an $r \times r$ matrix $M$ such that $M^\Phi_{A_\xi, A_\eta}(i, j) = M(i, j)$ for each $\{\xi, \eta\} \in [\Gamma]^2$ and $\{i, j\} \in [n]^2$. Now find $\Omega \in [\Gamma]^\lambda$ and $g: n \to p$ such that $g_\alpha(A_\alpha(i)) = g(i)$ for each $\alpha \in \Omega$. Let

$$q = \sum_{i \neq j} g(i) g(j) M(i, j) \quad \text{modulo } p.$$

If $q = 0$, define $d: r \to 2$ by $d(0) = 1$ and $d(i) = 0$ for $i \neq 0$ and let $d$ be constantly 0 if $q \neq 0$.

Theorem 6 ensures that there are $\xi$ and $\eta$ in $\Omega$ such that $\xi \in \eta$ and

$$M^\Phi_{A_\xi, A_\eta}(i, j) = \begin{cases} M(i, j) & \text{if } i \neq j, \\ d(i) & \text{if } i = j. \end{cases}$$

It follows that

$$[(m_\xi, g_\xi)(m_\eta, g_\eta)] = \begin{cases} (-q, 0) & \text{if } q \neq 0, \\ (g(0)^2, 0) & \text{if } q = 0, \end{cases}$$

contradicting the assumption that $(m_\xi, g_\xi)$ and $(m_\eta, g_\eta)$ commute.

It is now possible to choose $\theta(\xi)$ for each $\xi \in 2^\lambda$ such that $G_{X_\xi \setminus \theta(\xi)}$ is an extraspecial $p$-group with no abelian subgroup of size $\lambda$.

To see that $\xi \neq \eta$ implies that $G_{X_\xi \setminus \theta(\xi)}$ is not isomorphic to $G_{X_\eta \setminus \theta(\eta)}$ suppose that $F$ is such an isomorphism. Let $\{\alpha_\mu; \mu \in \lambda\}$ enumerate $(X_\xi \setminus \theta(\xi)) \setminus X_\eta$. Let $F((0, \chi_{\{\alpha_\mu\}})) = (m_\mu, g_\mu)$. As before we can use the delta system lemma to find $\Lambda \in [\lambda]^\lambda$, $k \in \omega$ and $R \in [X_\eta]^{<\aleph_0}$ such that $\{\text{support}(g_\mu); \mu \in \Lambda\}$ form a delta system with root $R$, $|\text{support}(g_\mu)| = k$ for $\mu \in \Lambda$ and such that the restriction of $g_\mu$ to $R$ is the same for all $\mu \in \Lambda$. By partitioning $\{\alpha_\mu; \mu \in \lambda\}$ into disjoint sets $A_\mu$ of size $p$ and reindexing we obtain $\{A_\mu; \mu \in \lambda\} \subseteq [(X_\xi \setminus \theta(\xi)) \setminus X_\eta]^p$, $t \in \omega$ and $\{(n, f_\mu); \mu \in \lambda\}$ such that $F((t, \chi_{A_\mu})) = (n, f_\mu)$ and $\mu \in \sigma$ implies $f_\mu < f_\sigma$.

Let $B_\mu = \text{support}(f_\mu)$ and $s = p \cdot k - |R| = |B_\mu|$. Furthermore it may be assumed there is $f: s \to p$ such that $f_\mu(B_\mu(i)) = f(i)$. Also note that $A_\mu$ is disjoint from $B_\mu$, for all $\mu$.

Now let $C_\mu = A_\mu \cup B_\mu$ and choose $\Gamma \in [\lambda]^\lambda$ such that $\{\mu, \sigma\} \in [\Gamma]^2$ and $\mu \in \sigma$ implies $\max C_\mu \in \min C_\sigma$ and such that there is $D \subseteq s + p$ such that $C_\mu(i) \in A_\mu$ if and only if $i \in D$ for every $\mu \in \Gamma$. As in the previous proof find $\Omega \in [\Gamma]^\lambda$ and a matrix $M$ such that $M^{\Phi}_{C_\gamma,C_\delta}(i,j)=M(i,j)$ if $\{\gamma,\delta\}\in[\Omega]^2$ and $i\neq j$. Let

$$x=\sum_{\{i,j\}\in[D]^2} M(i,j) \quad \text{modulo } p$$

and

$$y=\sum_{\{i,j\}\in[(s+p)\setminus D]^2} f(i)f(j)M(i,j) \quad \text{modulo } p.$$

Define $d: s+p\to 2$ by

$$d(i)=\begin{cases}1 & \text{if } i=D(j) \text{ and } j<x,\\ 0 & \text{if } i=D(j) \text{ and } j>x,\\ 0 & \text{if } i=((s+p)\setminus D)(j) \text{ and } j>0,\\ 0 & \text{if } i=((s+p)\setminus D)(j) \text{ and } j=0 \text{ and } y\neq 0,\\ 1 & \text{if } i=((s+p)\setminus D)(j) \text{ and } j=0 \text{ and } y=0.\end{cases}$$

Let $\bar{M}$ be the $(s+p)\times(s+p)$ matrix defined by

$$\bar{M}(i,j)=\begin{cases}M(i,j) & \text{if } i\neq j,\\ d(i) & \text{if } i=j.\end{cases}$$

Then find $\{\mu,\sigma\}\in[\Omega]^2$ such that $M^{\Phi}_{C_\mu,C_\sigma}=\bar{M}$. Then $(t,\chi_{A_\mu})$ and $(t,\chi_{A_\sigma})$ commute but $F((t,\chi_{A_\mu}))$ and $F((t,\chi_{A_\sigma}))$ do not.

It should be noted that it has actually been shown that $G_{X_\xi\setminus\theta(\xi)}$ is not even isomorphic to a subgroup of $G_{X_\eta\setminus\theta(\eta)}$. If the same construction is applied to a function $\Phi$ obtained from Theorem 5, then it can be shown that it is consistent that there is an extraspecial $p$-group of size $\aleph_2$ which does not even have an abelian subgroup of size $\aleph_1$.

## 4. Proof of Theorem 6

Let $\lambda$ be a cardinal satisfying the following conditions.

1. There is some $S\subseteq\lambda$ which is stationary but such that $S\cap\alpha$ is not stationary in $\alpha$ for every $\alpha\in\lambda$.
2. There is $\tau<\lambda$ such that $2^\tau\geqslant\lambda$.

For each limit ordinal $\alpha$ in $\lambda$, choose a closed unbounded subset of $\alpha$, $C_\alpha$, disjoint from $S$ and containing 0. If $\alpha=\beta+1$, let $C_\alpha=\{0,\beta\}$.

Now for $0\in\alpha\in\beta\in\lambda$ define $\Gamma^+_l(\alpha,\beta)$ and $\Gamma^-_l(\alpha,\beta)$ by induction on $l$ as follows:

(a) $\Gamma^+_0(\alpha,\beta)=\beta$ and $\Gamma^-_0(\alpha,\beta)=0$.
(b) If $\Gamma^+_l(\alpha,\beta)$ is defined and greater than $\alpha$, let

$$\Gamma^+_{l+1}(\alpha,\beta)=\min(C_{\Gamma^+_l(\alpha,\beta)}\setminus\alpha).$$

Define

$$\Gamma^-_{l+1}(\alpha,\beta)=\max(C_{\Gamma_l^+(\alpha,\beta)}\cap\alpha).$$

From this definition it follows $\alpha\leqslant\Gamma^+_{l+1}(\alpha,\beta)\in\Gamma^+_l(\alpha,\beta)$. Hence there is some least integer $k$ such that $\Gamma^+_k(\alpha,\beta)$ is not defined because $\Gamma^+_{k-1}(\alpha,\beta)=\alpha$. Let $k(\alpha,\beta)=k$. Let $\mu_m(\alpha,\beta)=\max\{\Gamma^-_l(\alpha,\beta)+1: l\in m\}$.

The following two facts will be used often in the rest of the proof: If $0\in\alpha\in\beta$ and $m\leqslant k(\alpha,\beta)$, then

(a) $\mu_m(\alpha,\beta)\leqslant\alpha$ and if $\alpha$ is a limit $\mu_m(\alpha,\beta)\in\alpha$,
(b) $\Gamma^+_l(\alpha,\beta)=\Gamma^+_l(\mu_m(\alpha,\beta),\beta)$ if $l\leqslant m$ and $\Gamma^-_l(\alpha,\beta)=\Gamma^-_l(\mu_m(\alpha,\beta),\beta)$ if $l\in m$.

Let $\{A_\alpha:\alpha\in\lambda\}$ be distinct subsets of $\tau$. Let $\{h_\alpha:\alpha\in\lambda\}$ list all functions from $\mathcal{P}(M)$ to $\lambda$ where $M\in[\tau]^{<\aleph_0}$ and let $\{S_\alpha:\alpha\in\lambda\}$ be a partition of $\lambda$ into stationary subsets. For $\zeta\in\lambda$ define $m(\zeta)$ as follows: Let $\theta$ be the unique ordinal such that $\zeta\in S_\theta$ and let $m(\zeta)$ be the finite subset of $\tau$ such that $\mathcal{P}(m(\zeta))$ is the domain of $h_\theta$.

The function $\Phi:[\lambda]^2\to\lambda$ will now be defined. Let $0\in\alpha\in\beta\in\lambda$. Let $i\leqslant k(\alpha,\beta)$ be maximal such that

(i) $\Gamma_l(\mu_i(\alpha,\beta),\alpha)=\Gamma^-_l(\mu_i(\alpha,\beta),\beta)$ for $l\in i$,
(ii) $A_\alpha\cap m(\Gamma^+_l(\mu_i(\alpha,\beta),\alpha))=A_\beta\cap m(\Gamma^+_l(\mu_i(\alpha,\beta),\beta))$ for $l\in i$,
(iii) $m(\Gamma^+_l(\mu_i(\alpha,\beta),\alpha))=m(\Gamma^+_l(\mu_i(\alpha,\beta),\beta))$.

If $i=0$, define $\Phi(\alpha,\beta)=0$. Otherwise define

$$\Phi(\alpha,\beta)=h_{\Gamma^+_{i-1}(\mu_i(\alpha,\beta),\beta)}(A_\beta\cap m(\Gamma^+_{i-1}(\mu_i(\alpha,\beta),\beta)).$$

It must now be shown that $\Phi$ has the required properties. Therefore suppose that $\{x_\zeta:\zeta\in\lambda\}\subseteq{}^n\lambda$ are one-to-one functions with disjoint ranges.

To see that the second property in the conclusion of Theorem 6 is satisfied, choose $m_\zeta\in[\tau]^{<\aleph_0}$ such that $|\{A_{x_\zeta(i)}\cap m_\zeta: i\in n\}|=n$ without loss of generality $m_\zeta=M$ and $A_{x_\zeta(i)}\cap M=a_i$ for $\zeta\in\lambda$ and $i\in n$. Now choose $\sigma$ such that the domain of $h_\sigma$ is $\mathcal{P}(M)$. Next, for $\delta\in S_\sigma$ choose $\theta(\delta)$ such that $(x''_{\theta(\delta)})\cap\delta=0$. Since $S_\sigma$ is stationary, there is $\beta\in\lambda$ and $s\in[S_\sigma]^\lambda$ such that $\mu(\delta,x_{\theta(\delta)}(i))\in\beta$ for $i\in n$ and $\delta\in S$. Let $\Sigma_\delta$ be the closure of $\{\delta\}\cup x''_{\theta(\delta)}n$ under $\Gamma^+_1$ and $\Gamma^-_1$. Notice that this closure is finite because the functions $\Gamma^+_1$ and $\Gamma^-_1$ are regressive in the second variable. Let $A_\delta$ be the function defined on $\Sigma_\delta$ by $A_\delta(\xi)=A_\xi\cap m(\xi)$. Let $H_\delta$ be defined by $H_\delta(\alpha,\beta)=h_\beta(A_\alpha\cap m(\beta))$. It follows that there is $\Lambda\in[S]^\lambda$ such that if $\{\zeta,\xi\}\subseteq\Lambda$ and $\zeta\in\xi$, then there is an isomorphism $I_{\zeta,\xi}$ of the two structures $(\Sigma_\zeta,\Gamma^+_1,\Gamma^-_1,A_\zeta,H_\zeta,\in)$ and $(\Sigma_\xi,\Gamma^+_1,\Gamma^-_1,A_\xi,H_\xi,\in)$ such that $I_{\zeta,\xi}$ is the identity below $\beta$. Moreover, it can be arranged that if $\zeta\in\xi$ and $\{\zeta,\xi\}\subseteq\Lambda$, then $\Sigma_\zeta\subseteq\xi$.

Now let $\{i,j\}\in[n]^2$ and $\zeta\in\xi$ such that $\{\zeta,\xi\}\subseteq\Lambda$. It will be shown that $\Phi(x_{\theta(\zeta)}(i),x_{\theta(\xi)}(j))$ depends only on $i,j$ and the unique isomorphism type of the structures indexed by $\Lambda$. First note that since $\beta \in x_{\theta(\zeta)}(i) \in \xi \in x_{\theta(\xi)}(j)$, it follows that $k(x_{\theta(\zeta)}(i), x_{\theta(\xi)}(j)) \geqslant k(\xi, x_{\theta(\xi)}(j))$. Now let $q \leqslant k(x_{\theta(\zeta)}(i), x_{\theta(\xi)}(j))$ be maximal satisfying (i), (ii) and (iii) in the definition of $\Phi$.

Now note that for $l \leqslant k(\xi, x_{\theta(\xi)}(j))$

$$\mu_l(x_{\theta(\zeta)}(i), x_{\theta(\xi)}(j)) = \mu_l(\xi, x_{\theta(\xi)}(j)) < \beta.$$

Hence the sequences

$$\{\Gamma_t^-(\mu_l(x_{\theta(\zeta)}(i), x_{\theta(\xi)}(j)), x_{\theta(\xi)}(j) : t \in l\},$$
$$\{\Gamma_t^-(\mu_l(x_{\theta(\zeta)}(i), x_{\theta(\xi)}(j)), x_{\theta(\zeta)}(i) : t \in l\},$$
$$\{A_{x_{\theta(\xi)}}(j) \cap m(\Gamma_t^+(\mu_l(x_{\theta(\zeta)}(i), x_{\theta(\xi)}(j)), x_{\theta(\xi)}(j)) : t \in l\},$$
$$\{A_{x_{\theta(\zeta)}}(i) \cap m(\Gamma_t^+(\mu_l(x_{\theta(\zeta)}(i), x_{\theta(\xi)}(j)), x_{\theta(\zeta)}(i)) : t \in l\},$$
$$\{m(\Gamma_t^+(\mu_l(x_{\theta(\zeta)}(i), x_{\theta(\xi)}(j)), x_{\theta(\xi)}(j) : t \in l\},$$
$$\{m(\Gamma_t^+(\mu_l(x_{\theta(\zeta)}(i), x_{\theta(\xi)}(j)), x_{\theta(\zeta)}(i) : t \in l\}$$

depend only on the unique isomorphism type of the structures indexed by $\Lambda$ if $l \leqslant k(\xi, x_{\theta(\xi)}(j))$. Since $\Phi(x_{\theta(\zeta)}(i), x_{\theta(\xi)}(j))$ is determined by these sequences and by the function $H_\xi$, it suffices to show that $q \leqslant k(\xi, x_{\theta(\xi)}(j))$.

To see this let $K = k(\xi, x_{\theta(\xi)}(j))$ and notice that $\Gamma_{K-1}^+(\mu_K(x_{\theta(\zeta)}(i), x_{\theta(\xi)}(j)), x_{\theta(\xi)}(j)) = \xi$ and $m(\xi) = M$. Then either $m(\Gamma_{K-1}^+(\mu_K(x_{\theta(\zeta)}(i), x_{\theta(\xi)}(j)), x_{\theta(\zeta)}(i))) \neq M$ in which case (iii) in the definition of $\Phi$ fails or else equality holds in which case

$$A_{x_{\theta(\zeta)}}(i) \cap m(\Gamma_{K-1}^+(\mu_K(x_{\theta(\zeta)}(i), x_{\theta(\xi)}(j)), x_{\theta(\zeta)}(i)))$$
$$= A_{x_{\theta(\zeta)}}(i) \cap M = a_i \neq a_j = A_{x_{\theta(\xi)}}(j) \cap m(\xi)$$

and so (ii) fails.

To see that every $\mathcal{M}_d$ is realized let $d : n \to \lambda$. As before it may be assumed without loss of generality that there are $m \in [\tau]^{<\aleph_0}$ and $\{a_i : i \in n\} \subseteq \mathcal{P}(m)$ such that $x_\zeta(i) \cap m = a_i$. Let $h : \mathcal{P}(m) \to \lambda$ be a function satisfying $h(a_i) = d(i)$ and suppose that $h_\sigma = h$.

Now let $\mathcal{H}$ be the structure $(H(\omega_2), \{x_\zeta : \zeta \in \lambda\}, \{C_\zeta : \zeta \in \lambda\})$ and let $\{\mathcal{N}_\delta : \delta \in \lambda\}$ be a continuous increasing sequence of small elementary submodels of $\mathcal{H}$ with $N_\delta$ the universe of $\mathcal{N}_\delta$. Since $S_\sigma$ is stationary it is possible to choose $\alpha \in \beta$ such that $\{\alpha, \beta\} \subseteq S_\sigma$ and $N_\alpha \cap \lambda = \alpha$ and $N_\beta \cap \lambda = \beta$.

Now $(x_\beta''n) \cap \beta = 0$. Let $K_j = k(\alpha, x_\beta(j))$, let $\gamma_i^j = \Gamma_i^-(\alpha, x_\beta(j))$ for $i \in K_j$. Let $m_{i,l}^j = m(\Gamma_l^+(\mu_i(\alpha, x_\beta(j)), x_\beta(j))$ and let $A_{i,l}^j = A_{x_\beta(j)} \cap m_{i,l}^j$. Then by elementarity

$$\mathcal{N}_\beta = \text{“}(\forall \rho \in \beta)(\exists \zeta \in \beta \setminus \rho)((x_\zeta''n) \cap \zeta = 0$$
$$\&(\forall j \in n)(k(\alpha, x_\zeta(j)) = K_j$$
$$\&(\forall i \in K_j)(\Gamma_i^-(\alpha, x_\zeta(j)) = \gamma_i^j$$
$$\&(\forall l \in i)(m(\Gamma_l^+(\mu_i(\alpha, x_\zeta(j)), x_\zeta(j))) = m_{i,l}^j$$
$$\&A_{x_\zeta(j)} \cap m_{i,l}^j = A_{i,l}^j))))\text{”}.$$

Since $\gamma_i^j \in \mathcal{N}_\alpha$ for every $i$ and $j$ it follows again by elementarity that

$$\begin{aligned}\mathcal{N}_\alpha = \text{“}&(\forall \mu \in \alpha)(\exists \theta \in \alpha\backslash\mu)(\forall \rho \in \alpha)(\exists \zeta \in \alpha\backslash\rho)((x_\zeta''n) \cap \zeta = 0\\ &\& \ (\forall j \in n)(k(\theta, x_\zeta(j)) = K_j\\ &\& \ (\forall i \in K_j)(\Gamma_i^-(\theta, x_\zeta(j)) = \gamma_i^j\\ &\& \ (\forall l \in i)(m(\Gamma_l^+(\mu_i(\theta, x_\zeta(j)), x_j(j))) = m_{i,l}^j\\ &\& \ A_{x_\zeta(j)} \cap m_{i,l}^j = A_{i,l}^j))))\text{”}.\end{aligned}$$

Now let $\mu = \max\{\mu_{K_j}(\alpha, x_\beta(j)) : j \in n\}$. Then $\mu \in \alpha$ and so there is $\theta \in \alpha\backslash\mu$ such that

$$\begin{aligned}\mathcal{N}_\alpha = \text{“}&(\forall \rho \in \alpha)(\exists \zeta \in \alpha\backslash\rho)((x_\zeta''n) \cap \zeta = 0\\ &\& \ (\forall j \in n)(k(\theta, x_\zeta(j)) = K_j\\ &\& \ (\forall i \in K_j)(\Gamma_i^-(\theta, x_\zeta(j)) = \gamma_i^j\\ &\& \ (\forall l \in i)(m(\Gamma_l^+(\mu_i(\tau, x_\zeta(j)), x_\zeta(j))) = m_{i,l}^j\\ &\& \ A_{x_\zeta(j)} \cap m_{i,l}^j = A_{i,l}^j))))\text{”}.\end{aligned}$$

Hence if $\delta \in C_\alpha$ is such that $\delta > \theta$, then there is $\zeta \in \alpha\backslash\delta$ such that the above statement holds. To see that $D(x_\zeta(j), x_\beta(j)) = G(j)$ let $k = k(\alpha, x_\beta(j))$. Then $k \leqslant k(x_\zeta(j), x_\beta(j))$ since $\mu_{k(\alpha, x_\beta(j))} < x_\zeta(j) < \alpha$. Also, $\Gamma_l^-(\alpha, x_\beta(j)) = \Gamma_l^-(x_\zeta(j), x_\beta(j))$ for $l \in m$. Hence $\mu_k(x_\zeta(j), x_\beta(j)) = \mu_k(\alpha, x_\beta(j)) = \mu_k(\tau, x_\zeta(j)) = \mu_k$. Therefore $\Gamma_l^-(\mu_k, x_\beta(j)) = \Gamma_l^-(\mu_k, x_\zeta(j))$ for $j \in m$ and also

$$\begin{aligned}&m(\Gamma_l^+(\mu_k, x_\beta(j)) = m(\Gamma_l^+(\mu_k, x_\zeta(j)) \quad \text{and}\\ &A_{x_\zeta(j)} \cap m(\Gamma_l^+(\mu_k, x_\zeta(j))) = A_{x_\beta(j)} \cap m(\Gamma_l^+(\mu_k, x_\beta(j)).\end{aligned}$$

Finally, note that $\Gamma_{k-1}^+(x_\zeta(j), x_\beta(j)) = \alpha$ and hence $\Gamma_k^-(x_\zeta(j), x_\beta(j)) \geqslant \delta > \theta$. Hence

$$\Gamma_k^+(\mu_{k+1}(x_\zeta(j), x_\beta(j)), x_\beta(j)) \geqslant \delta > \theta.$$

On the other hand

$$\Gamma_k^+(\mu_{k+1}(x_\zeta(j), x_\beta(j)), x_\zeta(j)) \geqslant \mu_{k+1}(x_\zeta(j), x_\beta(j)) \geqslant \delta > \theta$$

while

$$\Gamma_{k-1}^+(\theta, x_\zeta(j)) = \theta.$$

Hence there is some least $l \leqslant k$ such that

$$\Gamma_l^+(\theta, x_\zeta(j)) \neq \Gamma_l^+(\mu_{k+1}(x_\zeta(j), x_\beta(j)), x_\zeta(j)).$$

Then

$$\Gamma_{l-1}^+(\theta, x_\zeta(j)) = \Gamma_{l-1}^+(\mu_{k+1}(x_\zeta(j), x_\beta(j)), x_\zeta(j)) = \gamma = \Gamma_{l-1}^+(\alpha, x_\beta(j)).$$

But then

$$\begin{aligned}&\Gamma_l^-(\mu_{k+1}(x_\zeta(j), x_\beta(j)), x_\zeta(j)) + 1 = \Gamma_l^+(\mu_{k+1}(x_\zeta(j), x_\beta(j)), x_\zeta(j))\\ &\neq \Gamma_l^+(\mu_{k+1}(x_\zeta(j), x_\beta(j)), x_\beta(j)) = \Gamma_l^-(\mu_{k+1}(x_\zeta(j), x_\beta(j)), x_\beta(x_j)) + 1.\end{aligned}$$

Hence $\Gamma_l^-(\mu_{k+1}(x_\xi(j), x_\beta(j)), x_\xi(j)) \neq \Gamma_l^-(\mu_{k+1}(x_\xi(j), x_\beta(j)), x_\beta(j))$. Hence $k$ is maximal. Therefore

$$\begin{aligned} &D(x_\xi(j), x_\beta(j)) \\ &= h_{\Gamma_{k-1}^+(\mu_k(x_\xi(j), x_\beta(j)), x_\beta(j))} \cdots (A_{x_\beta(j)} \cap m(\Gamma_{k-1}^+(\mu_k(x_\xi(j), x_\beta(j)), x_\beta(j)))) \\ &= h(A_j) = G(j). \end{aligned}$$

## 5. Remarks

The last section showed that there is a proper class of uncountable regular cardinals $\lambda$ for which there is an Ehrenfeucht–Faber group of size $\lambda$. It should be noted that it is impossible to prove that there is such a group of every uncountable regular cardinality without assuming some extra set-theoretic hypothesis because of the following fact.

**Theorem 7.** *If $\kappa$ is a weakly compact cardinal and $G$ is a group of cardinality $\kappa$ such that $|G'| < \kappa$, $G' \subseteq Z(G)$ and $G'$ is periodic, then $G$ has an abelian subgroup of size $\kappa$.*

**Proof.** The following characterization of weak compactness will be used: $\kappa$ is weakly compact if and only if whenever the edges of the complete graph on $\kappa$ vertices are coloured with less than $\kappa$ colours, there is a complete subgraph of size $\kappa$ all of whose edges are the same colour.

First choose $\{a_\xi : \xi \in \kappa\} \subseteq G$ such that $a_\eta$ does not belong to the subgroup generated by $\{a_\xi : \xi \in \eta\}$. Think of the $a_\xi$ as vertices of a complete graph and colour the edge between $a_\xi$ and $a_\eta$ with $[a_\xi, a_\eta]$ where $\xi \in \eta$. From the weak compactness of $\kappa$ it follows that there is $\Gamma \in [\kappa]^\kappa$ and $C \in G'$ such that $[a_\xi, a_\eta] = C$ for $\{\xi, \eta\} \in [\kappa]^2$ satisfying $\xi \in \eta$. Since $C \in Z(G)$ it follows that $ga_\xi a_\eta^{-1}h = cga_\eta^{-1}a_\xi h$ when $\xi \in \eta$ and $g$ and $h$ are arbitrary elements of $G$. Repeated use of this identity yields that if $\xi_1 < \xi_2 < \cdots < \xi_{2k}$, then $[a_{\xi_1}a_{\xi_2}\cdots a_{\xi_k}, a_{\xi_{k+1}}a_{\xi_{k+2}}\cdots a_{\xi_{2k}}] = C^{k^2}$. If $m$ is the exponent of $C$ and $\{A_\xi : \xi \in \kappa\} \subseteq [\Gamma]^m$ are disjoint sets satisfying $\max A_\xi < \min A_\eta$ whenever $\xi \in \eta$, then let $b_\xi = a_{A_\xi(0)} \cdot a_{A_\xi(1)} \cdot \cdots \cdot a_{A_\xi(m-1)}$. Then $\{b_\xi : \xi \in \kappa\}$ generates an abelian subgroup of size $\kappa$.

Other conjectures are defeated by the following result which requires the next definition.

**Definition.** $\kappa \to [\lambda]^2_{m,n}$ means that if $F : [\kappa]^2 \to n$ there is $A \in [n]^m$ and $X \in [\kappa]^\lambda$ such that $F''[X]^2 \subseteq A$.

**Theorem 8.** *If there are $\kappa, \lambda, k$ such that for all finite $m \geqslant k$, $\kappa \to [\lambda]^2_{k,m}$, then every group $G$ of size $\kappa$ such that $G' \subseteq Z(G)$ and $|G'| < \aleph_0$ contains an abelian subgroup of size $\lambda$.*

**Proof.** First choose $\{a_\xi : \xi \in \kappa\} \subseteq G$ such that $a_\xi \notin \langle \{a_\eta : \eta \in \xi\} \rangle$. Then choose $\bar{m}$ so large that $\bar{m} \to (p)^2_{p^k}$ where $p = |G'|$. Let $m = p^k \bar{m}$ and let $b_\xi = a_{\bar{\xi}} \cdot a_{\bar{\xi}+1} \cdot a_{\bar{\xi}+2} \cdot \ldots \cdot a_{\bar{\xi}+m-1}$ where $\bar{\xi}$ is the $\xi$th limit ordinal in $\kappa$. Now define $h : [\kappa]^2 \to {}^{m^2}p$ by $h(\{\xi, \eta\})(i, j) = [a_{\bar{\xi}+i}, a_{\bar{\eta}+j}]$ where $\xi < \eta$. Since $\kappa \to [\lambda]^2_{k, p^{(m^2)}}$ there is $\Lambda \in [\kappa]^\lambda$ and $\{f_i \in {}^{m^2}p ; i \in k\}$ such that for all $\{\xi, \eta\} \subseteq \Lambda$ with $\xi < \eta$ there is $i \in k$ such that $[a_{\bar{\xi}+l}, a_{\bar{\xi}+l'}] = f_i(l, l')$ for all $l$ and $l'$. Since $m = p^k \bar{m}$ it follows that

$$(\exists X \in [m]^{\bar{m}})(\exists \sigma \in {}^k p)(\forall i \in X)(\forall j \in k)(\sigma(j) = f_j(i, i)).$$

Since $\bar{m} \to (p)^2_{p^k}$ it follows that

$$(\exists Y \in [X]^p)(\exists \theta \in {}^k p)(\forall i \in X)(\forall j \in X)(\forall l \in k)(f_l(i, j) = \theta(l)).$$

Let $d_\xi = \prod\{a_{\bar{\xi},1} : i \in Y\}$ for $\xi \in \Lambda$. Then for $\xi, \eta \in \Lambda$ with $\xi < \eta$ there is $l \in k$ such that $[a_{\bar{\xi}+l}, a_{\bar{\xi}+l'}] = f_i(l, l')$. Hence, letting $t = \max Y$,

$$\Big(\prod_Y a_{\bar{\xi},i}\Big)\Big(\prod_Y a_{\bar{\eta},i}\Big) = \Big(\prod_{j \in Y \setminus \{t\}} a_{\bar{\xi},j}\Big)\Big(\prod_Y a_{\bar{\eta},j}\Big)(a_{\bar{\xi},t})(f_i(5,5))\Big(\prod_{j \in Y \setminus \{t\}} f_i(jt, j)\Big)$$

$$= \Big(\prod_{j \in Y \setminus \{t\}} a_{\bar{\xi},j}\Big)\Big(\prod_Y a_{\bar{\eta},j}\Big)(a_{\bar{\xi},t})(\sigma(i))(\theta(i))^{p-1}.$$

Continuing in this way shows that

$$d_\xi \cdot d_\eta = d_\eta \cdot d_\xi(\sigma(i)(\theta(i))^{p-1})^p = d_\eta \cdot d_\xi.$$

In [5] it is shown that, for example, that it is consistent that $2^{\aleph_0} \to [\aleph_1]^2_{2,m}$ for all $m \in \omega$.

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